

# MANIFOLD STRUCTURES IN REGULAR IRREDUCIBLE ALGEBRAIC MONOIDS

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**ABSTRACT.** In this paper we study regular irreducible algebraic monoids over  $\mathbb{C}$  equipped with the euclidean topology. It is shown that, in such monoids, the Green classes and the spaces of idempotents in the Green classes all have natural manifold structures. The interactions of these manifold structures and the semigroup structures in these monoids have been investigated. Relations between these manifolds and Grassmann manifolds have been established. A generalisation of a result on the dimension of the manifold of rank  $k$  idempotents in the semigroup of linear endomorphisms over  $\mathbb{C}$  has been proved.

## 1. INTRODUCTION

Let  $K$  be an algebraically closed field. Let  $K[x_1, \dots, x_m]$  be the algebra of polynomials in the indeterminates  $x_1, \dots, x_m$  over  $K$ . A subset  $X$  of the affine  $m$ -space  $K^m$  is said to be algebraic if it is the zero set of a collection of polynomials in  $K[x_1, \dots, x_m]$  (Definition 1.2 [9]). The set  $X$  is irreducible if it is not a union of two proper algebraic sets. A (linear) algebraic semigroup  $S = (S, o)$  is an affine variety  $S$  along with an associative product map  $o : S \times S \rightarrow S$  which is also a morphism of varieties (Definition 3.1 [8]). Identifying the elements of the multiplicative semigroup  $M_n(K)$  of  $n \times n$  matrices over  $K$  with the elements of  $K^{n^2}$ , we can define algebraic sets in  $M_n(K)$ . By an algebraic subsemigroup of  $M_n(K)$  we mean a subsemigroup of  $M_n(K)$  which is also an algebraic subset of the affine  $n^2$ -space  $K^{n^2}$ . It is known that an algebraic semigroup  $S$  (respectively, an algebraic monoid  $M$ ) is isomorphic to an algebraic subsemigroup (respectively, an algebraic submonoid) of  $M_n(K)$  for some  $n$  (see Theorem 3.15, Corollary 3.16 [8]). So, in this paper, by an algebraic semigroup we shall always mean an algebraic subsemigroup of  $M_n(K)$  for some fixed  $n$ . Algebraic monoids have been extensively studied by Mohan S. Putcha [8], Lex E. Renner [9], and others.

A topological semigroup [2] is a Hausdorff space  $S$  together with an associative binary operation  $\cdot$  such that the product map  $(x, y) \mapsto x \cdot y$  is jointly continuous in  $x$  and  $y$ . The usual topology on an algebraic set is the Zarisky topology, for which the closed sets are its algebraic subsets. An algebraic set with the usual topology is not a Hausdorff space and so an algebraic semigroup cannot be a topological semigroup under the usual topology. However in the special case where  $K = \mathbb{C}$ , the field of complex numbers, there is another natural topology on algebraic sets, namely, the subspace topology inherited from the euclidean topology on  $M_n(\mathbb{C})$ . Since this topology is Hausdorff, an algebraic semigroup over  $\mathbb{C}$  is a topological semigroup under this topology (the joint continuity of the product map being obvious).

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It will be assumed throughout the rest of this paper that all algebraic semigroups are over the field  $\mathbb{C}$ . It will further be assumed that all topological terms refer to the euclidean topology. In this paper we study the topology of a regular irreducible algebraic monoid  $M$ . The special case where  $M = M_n(\mathbb{C})$  has been studied in [7]. We show here that many of the properties of  $M_n(\mathbb{C})$  can be generalised to  $M$ .

**Notations.** For notations and terminology relating to semigroups we have followed [1] and for those connected with topology of manifolds we have followed [4]. In particular the set of idempotents in a subset  $T$  of some semigroup will be denoted by  $E(T)$ . The letter  $M$  will always denote a regular irreducible algebraic submonoid of  $M_n(\mathbb{C})$  and  $G$  the group of units in  $M$ . We denote  $M_n(\mathbb{C})$  by  $M_n$ . Green's relations in  $M_n$  will be denoted by  $\mathcal{L}$ , etc. and the corresponding relations in  $M$  by  $\mathcal{L}^M$ , etc. The Green classes in  $M_n$  are denoted by  $L_a$ , etc. and the corresponding classes in  $M$  by  $L_a^M$ , etc. However if  $a \in M_n$  and  $\text{Rank}(a) = k$  we sometimes denote  $D_a$  by  $D_k$ .

In this paper we have considered several different (left) actions of groups on sets. These actions are denoted by  $\mathbf{A}_j$ , where  $j = 1, 2, \dots$ , the groups and the sets on which the groups act varying with  $j$ . Let  $\mathbf{A}_j : X \times Y \rightarrow Y$  denote the action of a group  $X$  on a set  $Y$ . The orbit of  $y \in Y$  under the action  $\mathbf{A}_j$  is  $\mathbf{A}_j(X, y)$ . The stabiliser of  $y \in Y$  under  $\mathbf{A}_j$  will be denoted by  $S_{j,y}$ . For a fixed element  $y \in Y$ , the natural map  $x \mapsto \mathbf{A}_j(x, y)$  from  $X$  into  $Y$  will be denoted by  $\mathbf{A}_{j,y}$ . The canonical map  $xS_{j,y} \mapsto \mathbf{A}_j(x, y)$  from  $X/S_{j,y}$  to  $\mathbf{A}_j(X, y)$  will be denoted by  $\phi_{j,y}$ .

## 2. PRELIMINARIES

We have the following elementary result describing the Euclidean topology of algebraic sets.

**Proposition 2.1.** *Let  $X \subseteq \mathbb{C}^n$  be an algebraic set. Then  $X$  is a Hausdorff, locally compact,  $\sigma$ -compact space.*

*Proof.* Since  $\mathbb{C}^n$  is Hausdorff,  $X$  is also Hausdorff. Since  $\mathbb{C}^n$  is locally compact and since every algebraic set in  $\mathbb{C}^n$  is closed,  $X$  must be locally compact.

For any positive integer  $r$  define  $V_r = \{x \in \mathbb{C}^n : \|x\| < r\}$  and  $U_r = X \cap V_r$ . Since  $\mathbb{C}^n = \bigcup_{r=1}^{\infty} V_r$ , we have  $X = \bigcup_{r=1}^{\infty} U_r$ . We also have  $\overline{U_r} = \overline{X \cap V_r} = X \cap \overline{V_r}$ . Now  $\overline{V_r}$  is compact. Since  $X$  is closed in  $\mathbb{C}^n$ ,  $\overline{U_r}$  is also compact in  $\mathbb{C}^n$ . Thus each  $U_r$  is relatively compact set in  $X$ . Obviously we also have  $\overline{U_r} \subseteq U_{r+1}$ . It follows (Theorem XI.7.2 [5]) that  $X$  is  $\sigma$ -compact.  $\square$

Since every algebraic semigroup is an algebraic set, we have the following corollary to Proposition 2.1.

**Corollary 2.2.** *Every algebraic semigroup is a Hausdorff, locally compact,  $\sigma$ -compact space.*

A subgroup  $A \subseteq \text{GL}(n)$  is said to be algebraic if  $A$  is an algebraic subset of  $\text{GL}(n)$ . We have the following result (Theorem 2.1.2 [10]) on algebraic groups.

**Theorem 2.3.** *Let  $A$  be an algebraic group in  $\text{GL}(n)$ . Then  $A$  is a closed analytic subgroup of  $\text{GL}(n)$ .*

In general,  $G$  is not an algebraic set in  $\text{GL}(n)$ . However we have the following result (see also Corollary 3.26 [8]).

**Lemma 2.4.**

- (1)  $G = M \cap \mathrm{GL}(n)$ .
- (2)  $G$  is an analytic subgroup of  $\mathrm{GL}(n)$ .

*Proof.* Clearly we have  $G \subseteq \mathrm{GL}(n)$  and so  $G \subseteq M \cap \mathrm{GL}(n)$ . Let  $u \in M \cap \mathrm{GL}(n)$ . Then  $\det(u) \neq 0$ . Hence (Remark 3.23 [8]) we must have  $u \mathcal{H}^M 1$ . But  $G = H_1^M$  and therefore  $u \in G$ . This proves the first half of the lemma.

$G$  is obviously a subgroup of  $\mathrm{GL}(n)$ . Hence, to prove that  $G$  is an analytic subgroup of  $\mathrm{GL}(n)$ , we need only show that  $G$  is a submanifold of  $\mathrm{GL}(n)$ . To prove this, we consider the set

$$(1) \quad G' = \left\{ \begin{bmatrix} \alpha & 0 \\ 0 & a \end{bmatrix} : \alpha \in \mathbb{K}, a \in M, \alpha \det(a) = 1 \right\}.$$

Since  $M$  is an algebraic set,  $G'$  is an algebraic set in  $\mathrm{GL}(n+1)$ .

Now, for any  $a \in G$ , we write  $a' = \begin{bmatrix} \frac{1}{\det(a)} & 0 \\ 0 & a \end{bmatrix}$ . Then  $G' = \{a' : a \in G\}$ . For  $a, b \in G$ , we have  $(ab)' = a'b' \in G'$  and so  $G'$  is a subgroup of  $\mathrm{GL}(n+1)$ . Thus  $G'$  is an algebraic subgroup of  $\mathrm{GL}(n+1)$ . Hence, by Theorem 2.3,  $G'$  is a closed analytic subgroup of  $\mathrm{GL}(n+1)$ .

The action of  $G'$  on the manifold  $\mathrm{GL}(n)$  defined by  $(a', u) \mapsto a' \cdot u = au$  is clearly an analytic left action. Hence the map  $\varsigma : G' \rightarrow G$ ,  $a' \mapsto a' \cdot 1 = a$  is a subimmersion. Obviously, the map  $\varsigma$  is injective and therefore ((16.8.8 (iv)) [4]) it is an immersion. Since the map  $a \mapsto \det(a)$  is continuous and open  $\varsigma$  is a homeomorphism. Since it is also an immersion, ((16.8.4) [4])  $G$  must be a submanifold of  $\mathrm{GL}(n)$ .  $\square$

The following theorems (see §16.10 [4]) are repeatedly used in the sequel.

**Theorem 2.5.** *Let an analytic group  $X$  act analytically on a manifold  $Y$ . Then the stabiliser  $S_y$  of  $y \in Y$  is an analytic subgroup of  $X$ .*

**Theorem 2.6.** *Let  $X$  be an analytic group acting analytically on a manifold  $Y$  under a map  $(x, y) \mapsto x \cdot y$ . If  $y \in Y$  is such that the orbit  $X \cdot y$  is locally closed in  $Y$  then  $X \cdot y$  is a submanifold of  $Y$  and the map  $\phi_x : X/S_y \rightarrow X \cdot y$  defined by  $xS_y \mapsto x \cdot y$  is an isomorphism of manifolds.*

### 3. TOPOLOGY OF GREEN CLASSES

We have the following characterizations of the Green classes in  $M$ . These follow from Proposition 6.1 [8], Theorem 1.4 [8] and Proposition II.4.5 [6].

**Lemma 3.1.** *Let  $a \in M$ . Then we have:*

- (1)  $L_a^M = Ga = L_a \cap M$ .
- (2)  $R_a^M = aG = R_a \cap M$ .
- (3)  $D_a^M = GaG = J_a^M$ .

The Green classes in  $M_n$  are submanifolds of  $M_n$ . We now prove that the result is true for  $M$  also.

**Proposition 3.2.** *Let  $a \in M$ . Then  $L_a^M$  and  $R_a^M$  are submanifolds of  $M_n$ .*

*Proof.* The map  $\mathbf{A}_1 : \mathrm{GL}(n) \times M_n \rightarrow M_n$  defined by  $(v, x) \mapsto vx$  is an analytic left action of  $\mathrm{GL}(n)$  on  $M_n$ . Since  $G$  is a submanifold of  $\mathrm{GL}(n)$  (see Lemma 2.4),  $G \times M_n$  is a submanifold of  $\mathrm{GL}(n) \times M_n$  and so the map

$$\mathbf{A}_1 : G \times M_n \rightarrow M_n, \quad (v, x) \mapsto vx,$$

which is the restriction of  $\mathbf{A}'_1$  to  $G$ , is also analytic ((16.8.3.4) [4]).

Now let  $a \in M$ . Lemma 3.1 implies that  $\mathbf{A}_1(G, a)$ , the orbit of  $a$  under  $\mathbf{A}_1$ , is  $L_a^M = L_a \cap M$ . It is known that  $L_a$  is a submanifold of  $M_n$  (Proposition 2.2 [7]). Therefore,  $L_a$  is a locally compact subspace of  $M_n$ . Since  $M_n$  is itself locally compact, we can find an open set  $U$  and a closed set  $F$  in  $M_n$  such that  $L_a = U \cap F$ . Therefore,  $L_a^M = U \cap (F \cap M)$ . Since  $M$  is an algebraic set in  $M_n$ , it is a closed set in  $M_n$ . Thus,  $L_a^M$  is the intersection of an open set and a closed set in  $M_n$ . It follows that  $L_a^M$  is locally closed in  $M_n$  ((12.2.3) [3]). Now, it follows that (see §16.10 [4]) that  $L_a^M$  is a submanifold of  $M_n$ .

To prove the result concerning  $R_a^M$  we consider the map

$$\mathbf{A}_2 : G \times M_n \rightarrow M_n, \quad (w, x) \mapsto xw^{-1}.$$

□

Since  $L_a$  and  $L_a^M$  are submanifolds of  $M_n$ , and since  $L_a^M \subseteq L_a$  (and similarly for  $R_a^M$ ) we have ((16.8.6.1) [4]):

**Corollary 3.3.**  $L_a^M$  is a submanifold of  $L_a$  and  $R_a^M$  is a submanifold of  $R_a$ .

Since  $L_a^M$  and  $R_a^M$  are locally closed subspaces of  $M_n$  we also have (see Theorem 2.6):

**Corollary 3.4.** For  $i = 1, 2$  we have:

- (1)  $S_{i,a}$  is an analytic subgroup of  $G$ .
- (2)  $\phi_{i,a}$  is an isomorphism of manifolds.
- (3)  $\mathbf{A}_{i,a}$  is a continuous open surjection.

We next consider the manifold structure of  $\mathcal{D}$ -classes of  $M$ .

**Lemma 3.5.** Let  $a \in M$ . Then  $D_a^M$  is a locally closed subspace of  $M_n$ .

*Proof.* By Lemma 3.1, we have  $D_a^M = GaG$ . Let  $F = \overline{MaM}$  be the Zarisky-closure of  $MaM$  in  $M_n$ . For any  $u \in G$ , the map  $y \mapsto uy$  is a Zarisky-homeomorphism of  $M_n$  onto itself. Therefore,  $uF = u\overline{MaM} = \overline{uMaM}$ . But, clearly, we have  $uM = M$  so that  $uMaM = MaM$ . Hence,  $uF = \overline{MaM} = F$ . Similarly, we also have  $Fu = F$ .

Now, there exists a Zarisky-open subset  $U_0$  of  $F$  such that  $U_0 \subseteq GaG$  (see proof of Proposition 6.1 in [8]). Since  $U_0$  is a Zarisky open subset of  $F$ , we can find a Zarisky-open set  $X_0$  in  $M_n(\mathbb{K})$  such that  $U_0 = F \cap X_0$ .

Choose some fixed element  $u_0av_0 \in U_0$ . Consider any element  $b = uav$  in  $GaG$ . We write  $U_b = u'U_0v'$  where  $u' = uu_0^{-1}$ ,  $v' = v_0^{-1}v$ . We show that  $U_b = F \cap (u'X_0v')$ .

Let  $x \in U_0$  then  $x \in F$  and  $x \in X_0$ . Since  $u'F = F$ , we have  $u'x \in F$ . Now  $u'x \in u'X_0$ . Therefore  $u'U_0 \subseteq F \cap (u'X_0)$ . To prove the inclusion in the reverse direction, let  $x \in F \cap (u'X_0)$ . Let  $x = u'x_0$  for some  $x_0 \in X_0$ . Then  $x_0 = (u')^{-1}x \in (u')^{-1}F = F$ . This shows that  $x = u'x_0 \in u'(F \cap X_0)$  implying that  $F \cap (u'X_0) \subseteq u'U_0$ . Hence  $F \cap (u'X_0) = u'U_0$ . By a similar argument we can now

show that  $F \cap (X_0 v') = U_0 v'$ . Now we have

$$\begin{aligned} U_b &= u' U_0 v' \\ &= (u' U_0) \cap (U_0 v') \\ &= (F \cap u' X_0) \cap (F \cap X_0 v') \\ &= F \cap (u' X_0 \cap X_0 v') \\ &= F \cap (u' X_0 v'). \end{aligned}$$

Since  $b \in U_b = F \cap u' X_0 v' \subseteq GaG$  we have

$$GaG = F \cap [\cup_{b \in GaG} (u' X_0 v')].$$

Since  $F$  is Zarisky-closed,  $F$  is a closed set in  $M_n$ . Since  $X_0$  is Zarisky-open it is open in  $M_n$ . Hence  $u' X_0 v'$  is also open. Therefore  $\cup_{b \in GaG} (u' X_0 v')$  is open in  $M_n$ . This shows that  $GaG$  is the intersection of a closed and an open set in  $M_n$ . Therefore  $GaG$  is a locally closed set in  $M_n$  ((12.2.3) [3]).  $\square$

**Proposition 3.6.** *Let  $a \in M$ . Then  $D_a^M$  is a submanifold of  $M_n$ .*

*Proof.* Consider the map

$$\mathbf{A}_3 : (G \times G) \times M_n \rightarrow M_n, \quad ((v, w), x) \mapsto vxw^{-1}.$$

This defines an analytic left action of  $G \times G$  on  $M_n$ . By Lemma 3.1,  $\mathbf{A}_3((G \times G), a) = GaG = D_a^M$ . Since  $GaG$  is a locally closed subspace of  $M_n$  (Lemma 3.5),  $D_a^M$  is a submanifold of  $M_n$  (Theorem 2.6).  $\square$

Since  $D_a$  is a submanifold of  $M_n$ , and since  $D_a^M \subseteq D_a$  we have:

**Corollary 3.7.**  *$D_a^M$  is a submanifold of  $D_a$ .*

Theorem 2.6 implies the following:

**Corollary 3.8.**  *$S_{3,a}$  is an analytic subgroup of  $G \times G$  and  $\phi_{3,a}$  is an isomorphism of manifolds.*

#### 4. GREEN CLASSES AND GRASSMANN MANIFOLDS

Let  $V$  be a fixed  $n$ -dimensional vector space over  $\mathbb{C}$  and let  $\mathbf{G}_k$  be the set of all  $k$ -dimensional subspaces of  $V$ . Choosing some fixed ordered basis  $B$  we can represent elements of  $V$  by coordinate vectors relative to  $B$ . Also elements of  $M_n$  can be looked upon as matrix representations of linear endomorphisms of  $V$  relative to  $B$ . If  $M(k, n; k)$  denotes the space of  $k \times n$  matrices of rank  $k$ , then the topology of  $\mathbf{G}_k$  is the quotient topology induced by the map  $q : M(k, n; k) \rightarrow \mathbf{G}_k$  which maps  $P \in M(k, n; k)$  to the subspace of  $V$  generated by the rows of  $P$ . This topology makes  $\mathbf{G}_k$  a manifold called the Grassmann Manifold. If  $x \in M \subseteq M_n$  and  $\text{Rank}(x) = k$  then the range  $\mathbf{R}(x)$  is in  $\mathbf{G}_k$  and the null space  $\mathbf{N}(x)$  is in  $\mathbf{G}_{n-k}$ .

**Lemma 4.1.** *Let  $a, b \in M$ . Then*

- (1)  $a \mathcal{L}^M b \Leftrightarrow \mathbf{R}(a) = \mathbf{R}(b)$ .
- (2)  $a \mathcal{R}^M b \Leftrightarrow \mathbf{N}(a) = \mathbf{N}(b)$ .
- (3)  $\mathcal{D}^M = \mathcal{J}^M$ .

*Proof.* Since  $M$  is a regular subsemigroup of  $M_n$ , for any  $a, b \in M$ ,  $a \mathcal{L}^M b$  if and only if  $a \mathcal{L} b$  (Proposition II.4.6 [6]). But  $a \mathcal{L} b$  if and only if  $R(a) = R(b)$  (see p.57 in [1]). This proves the claim regarding  $\mathcal{L}^M$ . The proof of the claim regarding  $\mathcal{R}^M$  is similar.

The result  $\mathcal{D}^M = \mathcal{J}^M$  follows from Theorem 1.4 [8].  $\square$

For  $a \in M$ , we write

$$(2) \quad G_a^l = \{R(x) : x \in D_a^M\}, \quad G_a^r = \{N(x) : x \in D_a^M\},$$

Note that if  $\text{Rank}(a) = k$  then  $G_a^l \subseteq G_k$  and  $G_a^r \subseteq G_{n-k}$ . We begin with the following lemma which shows that, in the definition of  $G_a^l$  [ $G_a^r$ ], we need consider only just one  $\mathcal{R}^M$ - [ $\mathcal{L}^M$ -] class contained in  $D_a^M$ .

**Lemma 4.2.** *If  $b \in D_a^M$  then*

- (1)  $G_a^l = \{R(x) : x \in R_b^M\}.$
- (2)  $G_a^r = \{N(x) : x \in L_b^M\}.$

*Proof.* Let  $W \in G_a^l$ . Then  $W = R(y)$  for some  $y \in D_a^M$ . Choose  $x \in R_b^M \cap L_y^M$ . Then  $R(x) = R(y) = W$  and so  $W \in \{R(x) : x \in R_b^M\}$ . Since the reverse inclusion is obvious we have the result concerning  $G_a^l$ . The proof of the result concerning  $G_a^r$  is similar.  $\square$

The main result of this section is that if  $\text{Rank}(a) = k$  then  $G_a^l$  [ $G_a^r$ ] is a submanifold of  $G_k$  [ $G_{n-k}$ ]. We begin with the following lemma.

**Lemma 4.3.** *Let The map  $\Gamma_k : D_k \rightarrow G_k$  [ $\Delta_k : D_k \rightarrow G_{n-k}$ ] defined by  $x \mapsto R(x)$  [ $x \mapsto N(x)$ ] is a continuous open surjection.*

*Proof.* Let  $U$  be an open set in  $G_k$ . Let  $x_0 \in \Gamma_k^{-1}(U)$ ,  $\Gamma_k(x_0) = W \in U$  and  $N(a) = N$ . Choose an ordered basis  $B$  for  $V$  such that the last  $(n-k)$ -vectors in  $B$  form an ordered basis for  $N(a)$ . Relative to  $B$ ,  $x_0 = \begin{bmatrix} X_0 \\ 0 \end{bmatrix}$  where  $X_0 \in M(k, n; k)$ . Since  $q^{-1}(U)$  is open in  $M(k, n; k)$  and  $X_0 \in q^{-1}(U)$  we can find  $\epsilon > 0$  such that

$$B_M(X_0, \epsilon) = \{X \in M(k, n; k) : \|X - X_0\| < \epsilon\} \subseteq q^{-1}(U).$$

Let  $B(x_0, \epsilon) = \{x \in D_k : \|x - x_0\| < \epsilon\}$ . If  $x \in B(x_0, \epsilon)$  and  $X$  is the matrix formed by the first  $k$  rows of  $x$  then  $\|X_0 - X\| < \epsilon$  so that  $X \in B_M(X_0, \epsilon)$ . Since  $X \in M(k, n; k)$ , the rank of  $X$  is  $k$ . Since  $\text{Rank}(x)$  is also  $k$ , the range  $R(x)$  of  $x$  is the subspace spanned by the rows of  $X$ . Thus we have  $R(x) = q(X)$ . Therefore,  $\Gamma_k(x) = R(x) = q(X) \in U$ . Hence we have  $B(x_0, \epsilon) \subseteq \Gamma_k^{-1}(U)$ . This shows that  $\Gamma_k^{-1}(U)$  is open. It follows that  $\Gamma_k$  is continuous.

To show that  $\Gamma_k$  is open, let  $U'$  be an open set in  $D_k$ . Now  $\Gamma_k^{-1}(\Gamma(U')) = \cup_{x \in U'} L_x$ . Also,  $y \in \cup_{x \in U'} L_x$  if and only if  $y \mathcal{L} x$  for some  $x \in U'$ . Also,  $y \mathcal{L} x$  if and only if  $y = ux$  for some  $u \in \text{GL}(n)$ . Thus,  $y \in \cup_{x \in U'} L_x$  if and only if  $y = ux$  for some  $x \in U'$  and for some  $u \in \text{GL}(n)$ . Therefore we have  $\Gamma_k^{-1}(\Gamma(U')) = \cup_{u \in \text{GL}(n)} uU'$ . The map  $x \mapsto ux$  is a homeomorphism of  $M_n$  onto itself. Since  $U'$  is open in  $D_k$ ,  $uU'$  is open in  $D_k$  for every  $u \in \text{GL}(n)$ . Therefore  $\Gamma_k^{-1}(\Gamma(U'))$  is a union of open sets in  $D_k$  and therefore is itself open. This shows that  $\Gamma_k$  is an open map.

In a similar way one can show that the map  $\Delta_k : D_k \rightarrow G_{n-k}$  defined by  $x \mapsto N(x)$  is also a continuous open surjection.  $\square$

**Proposition 4.4.** *The restrictions  $\Gamma_k|_{R_a}$  and  $\Delta_k|_{L_a}$  are continuous open surjections.*

*Proof.* For any open set  $U \subseteq R_a$ , we have  $\Gamma_k^{-1}(\Gamma_k(U)) = \cup_{x \in U} H_x$ . Now let  $e \in E(R_a)$ . For all  $c \in H_e$ , the map  $\lambda_c : x \mapsto cx$  is a bijection of  $R_a$  onto itself with inverse  $\lambda_{c'}$  where  $c'$  is the unique inverse of  $c$  in  $H_e$ . Since these translations are continuous the map  $\lambda_c$  is a homeomorphism and so  $\lambda_c(U) = cU$  is an open set in  $R_a$ . Further, since  $H_b = H_e b$  for all  $b \in R_a$ , we have  $\Gamma_k^{-1}(\Gamma_k(U)) = \cup_{c \in H_e} cU$  which is open in  $R_a$ . Since  $\Gamma_k$  is a quotient map,  $\Gamma_k(U)$  is open in  $G_k$ . This proves that  $\Gamma_k$  is an open map. Since  $\Gamma_k$  is a quotient map, clearly it is continuous. The proof is similar for  $\Delta|L_a$ .  $\square$

We now define the following maps:

$$\begin{aligned} (3) \quad & \Gamma_a^M : D_a^M \rightarrow G_a^l, \quad x \mapsto R(x), \\ (4) \quad & \Delta_a^M : D_a^M \rightarrow G_a^r, \quad x \mapsto N(x). \end{aligned}$$

Note that if  $a \in D_k$  then  $\Gamma_a^M = \Gamma_k|D_a^M$  and  $\Delta_a^M = \Delta_k|D_a^M$ . We now have the following result.

**Lemma 4.5.** *The map  $\Gamma_a^M [\Delta_a^M]$  is a continuous open surjection. Hence the topology of  $G_a^l [G_a^r]$  is the quotient topology induced by  $\Gamma_a^M [\Delta_a^M]$ .*

*Proof.* By Lemma 4.1, for any  $a, b \in M$  we have  $a \mathcal{L}^M b$  if and only if  $R(a) = R(b)$ . Hence the partition of  $D_a^M$  induced by  $\Gamma_a^M$  is the partition of  $D_a^M$  into  $\mathcal{L}$ -classes.

By Lemma 4.3, the map  $\Gamma_k$  is continuous. Hence its restriction to  $D_a^M$ , which is the map  $\Gamma_a^M$ , is also continuous. It is easy to see that if, in the proof of Lemma 4.3, in the proof of the fact that  $\Gamma_k$  is open we replace  $M_n(\mathbb{K})$  by  $M$  and  $\text{GL}(n)$  by  $G$ , then the resulting argument is still valid. It follows that the map  $\Gamma_a^M$  is open also.  $\square$

The analog of Proposition 4.4 is also valid. We state the result without proof.

**Lemma 4.6.** *The restriction  $\Gamma_a^M|R_a^M : R_a^M \rightarrow G_a^l [\Delta_a^M|L_a^M : L_a^M \rightarrow G_a^r]$  is a continuous open surjection. Therefore, the topology of  $G_a^l [G_a^r]$  is the quotient topology induced by this map.*

We now consider the map

$$\mathbf{A}_4 : G \times G_k \rightarrow G_k, \quad (u, W) \mapsto Wu^{-1}$$

which defines a left action of  $G$  on  $G_k$ . By introducing local coordinates in  $G_k$  we can see that this action is analytic. Theorem 2.5 implies that, for  $W \in G_k$ ,  $S_{4,W}$  is an analytic subgroup of  $G$ .

**Proposition 4.7.** *For  $e \in E(M)$ , we have  $S_{4,R(e)} = \{u \in G : eue = eu\}$ .*

*Proof.* Let  $u$  be an arbitrary element in  $G$ .

$$\begin{aligned} u \in S_{4,R(e)} &\Leftrightarrow R(e)u^{-1} = R(e) && \text{by definition of } S_W \\ &\Leftrightarrow R(e) = R(e)u \\ &\Leftrightarrow R(e) = R(eu) \\ &\Leftrightarrow e \mathcal{L} eu \\ &\Leftrightarrow ve = eu \text{ for some } v \in G \quad (\text{Lemma 3.1}) \end{aligned}$$

Now if  $eu = ve$  then we have  $eue = eu$ . Conversely, if  $eue = eu$  then  $e = e(ueu^{-1})$  which implies that  $e \mathcal{L}^M ueu^{-1}$  and so, by Lemma 3.1, we can find  $w \in G$  such

that  $e = w(ueu^{-1})$ . Taking  $v = wu$  we get  $eu = ve$ . Therefore  $S_{4,R(e)} = \{u \in G : eue = eu\}$ .  $\square$

In the terminology and notations of [8] (see p.48 [8]),  $S_{4,R(e)}$  is the *left centralizer*  $C_G^l(e)$  of  $e$  in  $G$ .

**Theorem 4.8.** *Let  $a \in M$ ,  $\text{Rank}(a) = k$ . Then  $G_a^l$  is a submanifold of  $G_k$ .*

*Proof.* Let  $W_0 = R(a)$ . Then  $S_{4,W_0}$  is an analytic subgroup of  $G$ . Hence the orbit manifold  $G/S_{4,W_0}$  exists (see (16.10.6) in [4]).

$$\begin{array}{ccccc}
 G & \xrightarrow{\mathbf{A}_{2,a}} & R_a^M & & \\
 \pi \downarrow & \searrow \mathbf{A}_{4,W_0} & \downarrow \Gamma_a^M & & \\
 G/S_{4,W_0} & \xrightarrow{\phi_{4,W_0}} & G_a^l & \xrightarrow{\subseteq} & G_k
 \end{array}$$

Let  $\pi : G \rightarrow G/S_{4,W_0}$  be defined by  $u \mapsto uS_{4,W_0}$ . Then we have  $\mathbf{A}_{4,W_0} = \pi \circ \phi_{4,W_0}$ .

Since  $\mathbf{A}_4$  is analytic action of the analytic group  $G$  on the analytic manifold  $G_k$ , the map  $\mathbf{A}_{4,W_0}$  is a subimmersion (see (16.10.2) in [4]). Therefore, the map  $\phi_{4,W_0}$  is also a subimmersion (see (16.10.4) in [4]). Since  $\phi_{4,W_0}$  is injective also, it is an immersion (see (16.8.8(iv)) in [4]).

Now  $\phi_{4,W_0}(G/S_{4,W_0}) = G_a^l$ . Hence, to show that  $G_a^l$  is a submanifold of  $G_k$  it is enough to show that (see (16.8.4) in [4]) the map  $\phi_{4,W_0}$  is a homeomorphism from  $G/S_{4,W_0}$  onto the subspace  $G_a^l$  of  $G_k$ .

To prove that  $\phi_{4,W_0}$  is a homeomorphism, we note that, we have  $\mathbf{A}_{4,W_0} = \mathbf{A}_{2,a} \circ \Gamma_a^M$ .

By Lemma 4.1,  $\Gamma_a^M$  is a continuous open surjection from  $G$  onto  $R_a^M$ . By Corollary 3.4 the map  $\mathbf{A}_{2,a}$  is a continuous open surjection from  $R_a^M$  onto  $G_a^l$ . Therefore the map  $\mathbf{A}_{4,W_0}$  is also a continuous open surjection from  $G$  onto  $G_a^l$ . Hence the topology on  $G_a^l$  is the quotient topology induced by the map  $\mathbf{A}_{4,W_0}$ . The quotient space induced by this map is  $G/S_{4,W_0}$ . Hence  $\phi_{4,W_0} : G/S_{4,W_0} \rightarrow G_a^l$ , which is the quotient map induced by the map  $\mathbf{A}_{4,W_0}$ , must be a homeomorphism.

The proof of the theorem is now complete.  $\square$

From Proposition 4.7 we have the following corollary (see (16.8.4) in [4]).

**Corollary 4.9.** *Let  $e \in E(M)$ . Then  $C_G^l(e)$  is an analytic subgroup of  $G$  and  $\phi_{4,R(e)}$  is an isomorphism of manifolds.*

We have corresponding results involving  $\mathcal{L}$ -classes. In this case we consider the action

$$\mathbf{A}_5 : G \times G_{n-k} \rightarrow G_{n-k}, \quad (u, N) \mapsto Nu^{-1}.$$

## 5. THE SPACE OF IDEMPOTENTS

We begin with the following lemma which characterizes the restrictions of Green's relations to the set of idempotents in  $M$  (see Corollary 6.8 in [8]). Note that the restrictions of  $\mathcal{L}$  and  $\mathcal{R}$  to  $E(M)$  are the biorder relations  $L$  and  $R$  in  $E(M)$ .



**Lemma 5.1.** *Let  $e, f \in E(M)$ . Then we have:*

- (1)  $e \mathcal{D}^M f \Leftrightarrow f = ueu^{-1}$  for some  $u \in G$ .
- (2)  $e \mathcal{L}^M f \Leftrightarrow f = ueu^{-1} = ue$  for some  $u \in G$ .
- (3)  $e \mathcal{R}^M f \Leftrightarrow f = ueu^{-1} = eu^{-1}$  for some  $u \in G$ .

Lemma 5.1 has the following consequence.

**Proposition 5.2.** *Any two  $\mathcal{L}^M$ - [ $\mathcal{R}^M$ -] classes contained in the same  $\mathcal{D}^M$ -class of  $M$  are homeomorphic under a conjugation. Moreover, this homeomorphism preserves the idempotents.*

*Proof.* Let  $b, c \in D_a^M$  and let  $f \in E(L_b^M)$  and  $g \in E(L_c^M)$ . Since  $f \mathcal{D}^M g$ , by Lemma 5.1, we can find  $u \in G$  such that  $g = ufu^{-1}$ .

Now, let  $x \in L_b^M$  and  $y = xux^{-1}$ . Since  $x \mathcal{L}^M f$ , by Lemma 3.1, we can find  $v \in G$  such that  $x = vf$ . Now we have

$$y = xux^{-1} = uvfu^{-1} = uvu^{-1}(ufu^{-1}) = (uvu^{-1})g.$$

This shows that  $y \mathcal{L}^M g$  and so  $y \mathcal{L}^M c$ . Conversely, if  $y \in L_c^M$  then it can be shown that  $u^{-1}yu \in L_b^M$ . Therefore the map  $x \mapsto xux^{-1}$  is a bijection from  $L_b^M$  onto  $L_c^M$ . This map is clearly a homeomorphism. It is also a conjugation. Thus  $L_b^M$  and  $L_c^M$  are homeomorphic under a conjugation. This homeomorphism obviously preserves idempotents.

The proof of the result regarding  $\mathcal{R}^M$ -classes is similar.  $\square$

Let  $e \in E(M)$ . To discuss the topology of  $E(D_e^M)$ , we consider the action  $\mathbf{A}_6$  of  $G$  on  $E(D_e^M)$  defined by

$$\mathbf{A}_6 : G \times E(D_e) \rightarrow E(D_e), \quad (u, f) \mapsto ufu^{-1}.$$

As in the proof of Proposition 3.2, we can see that this action is analytic and, by Lemma 5.1, we have  $E(D_e^M) = \mathbf{A}_6(G, e)$ . We also have

$$S_{6,e} = \{u \in G : ueu^{-1} = e\} = \{u \in G : ue = eu\}.$$

In the notations and terminology of [8] (see p.48 [8]),  $S_{6,e}$  is the *centraliser*  $C_G(e)$  of  $e$  in  $G$ .

**Proposition 5.3.** *Let  $e \in E(M)$ . Then  $E(D_e^M)$  is a submanifold of  $E(D_e)$ . Moreover, the map  $\phi_{6,e}$  is an isomorphism of manifolds.*

*Proof.* By Lemma 4.1 we have  $\mathcal{D}^M = \mathcal{J}^M$ . Hence Proposition 5.8 in [8] implies that  $E(D_e^M)$  is an irreducible algebraic subset of  $M$  and hence it is itself an algebraic set. Therefore  $E(D_e^M)$  is a closed set in  $M_n(\mathbb{K})$ . Since  $E(D_e)$  is a closed set in  $M_n(\mathbb{K})$  (see [7]) and since  $E(D_e^M)$  is also a closed set in  $M_n(\mathbb{K})$ ,  $E(D_e^M)$  is a closed set in  $E(D_e)$ . Hence  $E(D_e^M)$  is a locally closed set in  $E(D_e)$ .

Since  $E(D_e^M) = G \cdot e$  under the action  $\mathbf{A}_6$  of  $G$  on  $E(D_e)$ , Theorem 2.6 implies that  $E(D_e^M)$  is a submanifold of  $E(D_e)$ .

Since  $G$  is an analytic group and since  $\mathbf{A}_6$  is an analytic action on the manifold  $E(D_e)$ , the stabiliser  $C_G(e)$  is an analytic subgroup of  $G$ . Theorem 2.6 also implies that the map  $\phi_{6,e}$  is an isomorphism of manifolds.  $\square$

Since  $E(D_a^M)$  is a closed set in  $M$ , it is a closed set in  $E(M)$  also. The family of  $\mathcal{D}^M$ -classes in  $M$  is a finite family (see Theorem 5.10 in [8]). So the family  $\{E(D_e^M) : e \in E(M)\}$ , the union of whose members is  $E(M)$ , contains only a finite number of mutually disjoint, closed subsets of  $E(M)$ .

Let  $e, f \in E(M)$  and  $\text{Rank}(e) = \text{Rank}(f)$ . Then  $E(D_e) = E(D_f)$ . But  $E(D_e^M)$  and  $E(D_f^M)$  need not be homeomorphic. They need not even be manifolds of the same dimension. For example, consider the monoid

$$M = \left\{ \begin{bmatrix} \alpha & 0 \\ 0 & A \end{bmatrix} : \alpha \in \mathbb{K}, A \in M_2(\mathbb{K}) \right\}.$$

The group of units of this monoid is

$$G = \left\{ \begin{bmatrix} \alpha & 0 \\ 0 & A \end{bmatrix} : \alpha \det(A) \neq 0 \right\}.$$

Let  $e = \begin{bmatrix} 1 & 0 \\ 0 & O' \end{bmatrix}$  where  $O' \in M_2(\mathbb{K})$  is the zero-matrix and  $f = \begin{bmatrix} 0 & 0 \\ 0 & f' \end{bmatrix}$  where  $f' = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \in M_2(\mathbb{K})$ . Then  $e, f \in E(M)$  and  $\text{Rank}(e) = \text{Rank}(f) = 2$ . Now we can easily verify that

$$E(D_e^M) = \{e\} \quad \text{and} \quad E(D_f^M) = \left\{ \begin{bmatrix} 0 & 0 \\ 0 & g' \end{bmatrix} : g' \in E(D_{f'}) \subseteq M_2(\mathbb{K}) \right\}.$$

Obviously  $\dim E(D_e^M) = 0$ . Since  $\text{Rank}(f') = 1$ , by Theorem 5.3 [7], we have  $\dim E(D_{f'}) = 1$  and  $E(D_{f'}) = E(D_{f'})$ .

It is known that if  $e \in E(M)$  and  $\text{Rank}(e) = k$ , then  $E(L_e)$  and  $E(R_e)$  are affine spaces of dimension  $k(n - k)$  (Proposition 3.1[7]). We show here that, if  $e \in M$ , then  $E(L_e^M)$  and  $E(R_e^M)$  are submanifolds of these affine spaces. We begin with the following lemma.

**Lemma 5.4.** *If  $e, f \in E(M)$  and  $e \mathcal{L}^M f$  then  $E(L_e^M) = \{ufu^{-1} : u \in C_G^l(e)\}$ .*

*Proof.* Since  $e \mathcal{L}^M f$ , by Lemma 4.1,  $R(e) = R(f)$  and so by Proposition 4.7 we have  $C_G^l(e) = C_G^l(f)$ .

Let  $u \in C_G^l(e)$  and  $g = ufu^{-1}$ . We show that  $g \in E(L_e^M)$ .  $g$  is clearly in  $E(M)$ . Since  $C_G^l(e) = C_G^l(f)$  we have  $ufu = fu$ . Since  $C_G^l(f)$  is a group,  $u^{-1} \in C_G^l(f)$  and so  $fu^{-1}f = fu^{-1}$ . From these we get  $f(ufu^{-1}) = f$  and  $(ufu^{-1})f = ufu^{-1}$ , that is,  $fg = f$  and  $gf = g$ . Therefore  $g \mathcal{L}^M f$ .

Let  $g$  be an arbitrary element in  $E(L_e^M)$ . Then  $g \mathcal{L}^M f$ , and by Lemma 5.1, we can find  $u \in G$  such that  $g = ufu^{-1} = uf$ . For this  $u$ , we have  $f(ufu^{-1}) = fg = f$  and so  $ufu = fu$  implying that  $u \in C_G^l(f) = C_G^l(e)$ . Thus  $g = ufu^{-1}$  for some  $u \in C_G^l(e)$ .  $\square$

We next consider the following action of  $C_G^l(e)$  on  $E(L_e)$ .

$$\mathbf{A}_7 : C_G^l(e) \times E(L_e) \rightarrow E(L_e), \quad (u, f) \mapsto ufu^{-1}.$$

**Lemma 5.5.** *Let  $e \in E(M)$ .*

- (1) *The action  $\mathbf{A}_7$  is analytic.*
- (2)  *$S_{7,e} = C_G(e)$ .*
- (3)  *$\mathbf{A}_7(C_G^l(e), e) = E(L_e^M)$ .*

*Proof.* That  $\mathbf{A}_7$  is analytic is obvious. Obviously  $S_{7,e} \subseteq C_G(e)$ . If  $u \in C_G(e)$  then  $ueu^{-1} = e$  and so  $eue = eu$ , and hence  $u \in C_G^l(e)$ . Therefore  $u \in S_{7,e}$  implying that  $C_G(e) \subseteq S_{7,e}$ . Item (3) follows from Lemma 5.4.  $\square$

**Theorem 5.6.** *Let  $e \in E(M)$ .*

- (1)  *$E(L_e^M)$  is a submanifold of  $E(L_e)$  and also of  $E(D_e^M)$ .*
- (2)  *$C_G(e)$  is an analytic subgroup of  $C_G^l(e)$ .*

(3)  $\phi_{7,e}$  is an isomorphism of manifolds.

*Proof.* It is known that  $E(L_e^M)$  is an irreducible algebraic subset of  $M$  (Proposition 5.8 [8]). Hence it is itself an algebraic set. So, by Lemma 2.1, it is a locally compact space and hence it is locally closed also. Since  $E(L_e^M)$  is the orbit of  $e$  under the action  $\mathbf{A}_7$ , by Theorem 2.6,  $E(L_e^M)$  is a submanifold of  $E(L_e)$ . Theorem 2.6 also implies that  $\phi_{7,e}$  is an isomorphism of manifolds.  $\square$

We have a corresponding result involving  $\mathcal{R}$ -classes. In this case we consider the action

$$\mathbf{A}_8 : C_G^r(e) \times E(R_e) \rightarrow E(R_e), \quad (u, f) \mapsto ufu^{-1}.$$

We have some results involving the dimensions of the various manifolds considered in this section.

**Theorem 5.7.** *Let  $e \in E(M)$ . Then*

- (1)  $\dim E(L_e^M) = \dim \mathbf{G}_e^r$ .
- (2)  $\dim E(R_e^M) = \dim \mathbf{G}_e^l$ .

*Proof.* Let  $\mathbf{R}(e) = W$ . Then  $U_W = \{\mathbf{R}(f) : f \in E(R_e)\}$  is an open set in  $\mathbf{G}_k$  and the map  $\Gamma_k^e : E(R_e) \rightarrow U_W$  defined by  $f \mapsto \mathbf{R}(f)$  is a homeomorphism (see Theorem 4.4 [7]).

Next, let  $U_W^M = \{\mathbf{R}(f) : f \in E(R_e^M)\}$ . Then the restriction  $\Gamma_k^e|E(R_e^M)$  is a homeomorphism of  $E(R_e^M)$  onto  $U_W^M$ .

We now show that  $U_W^M = U_W \cap \mathbf{G}_e^l$  so that  $U_W^M$  is an open set in  $\mathbf{G}_e^l$ . Let  $W' \in U_W \cap \mathbf{G}_e^l$ . Then  $W' = \mathbf{R}(f) = \mathbf{R}(a)$  for some  $f \in E(R_e), a \in R_e^M$ . Since  $\mathbf{R}(f) = \mathbf{R}(a)$ , we have  $f \mathcal{L} a$  in  $M_n(\mathbb{K})$ . Since  $a \mathcal{R} e$  in  $M$  we have  $a \mathcal{R} e$  in  $M_n(\mathbb{K})$  (see Proposition II.4.5 [6]). We also have  $f \mathcal{R} e$  in  $M_n(\mathbb{K})$ . Thus  $f \mathcal{H} a$  in  $M_n(\mathbb{K})$ . Since  $f$  is an idempotent,  $H_a$  is a subgroup of  $M_n(\mathbb{K})$  with identity  $f$ . Therefore  $H_a^M$  is a subgroup of  $M$  and  $H_a^M \subseteq H_a$ . This implies that  $f \in H_a^M$  and so  $f \mathcal{R} a \mathcal{R} e$  in  $M$ . Hence  $W' = \mathbf{R}(f) \in U_W^M$ . Therefore  $U_W \cap \mathbf{G}_e^l \subseteq U_W^M$ . The reverse inclusion is obvious.

Therefore  $\Gamma_k^e|E(R_e^M)$  is a homeomorphism of  $E(R_e^M)$  onto an open set in  $\mathbf{G}_e^l$ . From this it follows that  $\dim E(R_e^M) = \dim \mathbf{G}_e^l$ . The proof of the other result is similar.  $\square$

If  $e \in E(M_n)$  then we have (see Proposition 3.1, Theorem 5.3 [7])

$$\dim E(D_e) = \dim E(L_e) + \dim E(R_e).$$

The next theorem is a generalization of this result.

**Theorem 5.8.** *Let  $e \in E(M)$ . Then*

$$\dim E(D_e^M) = \dim E(L_e^M) + \dim E(R_e^M).$$

*Proof.* Let

$$\begin{aligned} Z_e^M &= \{(\mathbf{R}(f), \mathbf{N}(f)) : f \in E(D_e^M)\}, \\ Z' &= \{(W, N) \in \mathbf{G}_a^l \times \mathbf{G}_a^r : W \oplus N = V\}. \end{aligned}$$

We first show that  $Z_e^M = Z'$ .

Obviously  $Z_e^M \subseteq Z'$ . Conversely, let  $(W, N) \in Z'$ . Let  $a, b \in D_e^M$  be such that  $\mathbf{R}(a) = W, \mathbf{N}(b) = N$  and consider  $H = L_a^M \cap R_b^M$ . Let  $x, y \in H$ . Now  $\mathbf{R}(x) = \mathbf{R}(y) = W$  and  $\mathbf{N}(x) = \mathbf{N}(y) = N$  and  $W \oplus N = V$ . Hence  $\mathbf{R}(xy) = W$  and

$N(xy) = N$ . By Lemma 4.1 we now have  $xy \in L_a^M \cap R_b^M = H$ . Therefore  $H$  is a subgroup of  $M$  (see Theorem 2.16 [1]). If  $g$  is the idempotent in  $H$  then, obviously,  $g \in E(D_e^M)$  and  $W = R(g)$ ,  $N = N(g)$ . This proves that  $(W, N) \in Z_e^M$  showing that  $Z' \subseteq Z_e^M$ .

Now, let  $\text{Rank}(e) = k$  and

$$Z_k = \{(W, N) \in G_k \times G_{n-k} : W \oplus N = V\}.$$

By Theorem 5.6 [7],  $Z_k$  is an open subset of  $G_k \times G_{n-k}$  and the map  $\zeta : f \mapsto (R(f), N(f))$  is a homeomorphism of  $E(D_e)$  onto  $Z_k$ . The restriction  $\zeta|_{E(D_e^M)}$  is a homeomorphism of  $E(D_e^M)$  onto  $Z_e^M$ .

Since  $G_a^l \subseteq G_k$  and  $G_a^r \subseteq G_{n-k}$  the topology on the product space  $G_a^l \times G_a^r$  is the subspace topology inherited from the product space  $G_k \times G_{n-k}$ . Since  $Z_k$  is open in  $G_k \times G_{n-k}$  and since  $Z_e^M = Z_k \cap (G_a^l \times G_a^r)$ ,  $Z_e^M$  is an open set in  $G_a^l \times G_a^r$ . Thus  $E(D_e^M)$  is homeomorphic to an open set in  $G_a^l \times G_a^r$ . From this we have

$$\dim E(D_e^M) = \dim G_a^l + \dim G_a^r.$$

The theorem now follows from Theorem 5.7.  $\square$

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